

# INEQUALITIES FOR THE HODGE NUMBERS OF IRREGULAR COMPACT KÄHLER MANIFOLDS

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**ABSTRACT.** Based on work of R. Lazarsfeld and M. Popa, we use the derivative complex associated to the bundle of the holomorphic  $p$ -forms to provide inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. For 3-folds and 4-folds we give an asymptotic bound for all the Hodge numbers in terms of the irregularity. As a byproduct, via the BGG correspondence, we also bound the regularity of the exterior cohomology modules of bundles of holomorphic  $p$ -forms.

## 1. INTRODUCTION

Given an irregular compact Kähler manifold  $X$ , a problem one tries to understand is under which hypotheses there exist relations between its various Hodge numbers  $h^{p,q}(X)$ . Along these lines, one can ask whether there exist formulas for the  $h^{p,q}(X)$ 's in terms of the fundamental invariant  $q(X) = h^{1,0}(X)$ , the *irregularity* of  $X$ . A classical result in this direction is the Castelnuovo-De Franchis inequality  $h^{0,2}(X) \geq 2q(X) - 3$  (see [BHPV] IV.5.2), which holds for surfaces that do not carry any fibrations onto a smooth curve of genus  $g \geq 2$ . This was generalized by F. Catanese to higher dimensional manifolds as follows: if a manifold  $X$  does not admit any higher irrational pencil<sup>1</sup> then  $h^{0,k}(X) \geq k(q(X) - k) + 1$  for all  $k$  (*cf.* [Cat]), by means of sophisticated arguments involving the exterior algebra of holomorphic forms. Another generalization of the Castelnuovo-De Franchis inequality is provided by the work of G. Pareschi and M. Popa. Theorem A in [PP2] states that if  $X$  is of maximal Albanese dimension and does not carry any higher irrational pencil then  $\chi(\omega_X) \geq q(X) - \dim X$ . Their inequality is deduced using Generic Vanishing Theory for irregular varieties and the Evans-Griffith Syzygy Theorem.

New techniques for the study of this problem were introduced recently by R. Lazarsfeld and M. Popa in [LP]. Their approach relies on the study of a global version of the *derivative complex associated to the structure sheaf*  $\mathcal{O}_X$  and on the theory of vector bundles on projective spaces. Their inequalities mainly involve Hodge numbers of type  $h^{0,k}(X)$ .

In this paper we extend the methods of [LP] to the bundles of holomorphic  $p$ -forms  $\Omega_X^p$ . In this way we get inequalities for all the Hodge numbers of a special class of irregular compact Kähler manifolds. The main idea behind the inequalities of [LP] and the ones of

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<sup>1</sup>A *higher irrational pencil* is a surjective map with connected fibers  $f : X \rightarrow Y$  having the property that any resolution singularities of  $Y$  is of maximal Albanese dimension and with non-surjective Albanese map.

this paper goes as follows. Via cup product any element  $0 \neq v \in H^1(X, \mathcal{O}_X)$  defines a complex of vector spaces

$$(1) \quad 0 \longrightarrow H^0(X, \Omega_X^p) \xrightarrow{\cup v} H^1(X, \Omega_X^p) \xrightarrow{\cup v} \dots \xrightarrow{\cup v} H^d(X, \Omega_X^p) \longrightarrow 0.$$

Denoting by  $\mathbf{P} = \mathbf{P}_{\text{sub}}(H^1(X, \mathcal{O}_X))$  the projective space of one dimensional linear subspaces of  $H^1(X, \mathcal{O}_X)$ , we can arrange all of these, as  $v$  varies, into a complex of locally free sheaves on  $\mathbf{P}$ :

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0,$$

whose fiber at a point  $[v] \in \mathbf{P}$  is the complex (1). The complex (1) is known as the *derivative complex associated to the bundle  $\Omega_X^p$  with respect to the vector  $v$* . It was introduced for the first time by M. Green and R. Lazarsfeld in [GL1] and [GL2] in their study of Generic Vanishing Theorems for irregular compact Kähler manifolds. At this point it is important to study the exactness of the complex (2). In [LP] the case  $p = 0$  is analyzed. For this case we have that if  $X$  does not carry any irregular fibrations, i.e. morphisms  $f : X \longrightarrow Y$  with connected fibers having the property that any smooth model of  $Y$  is of maximal Albanese dimension, then the complex (2) is everywhere exact except possibly at the last term and furthermore all the involved maps are of constant rank. In general, for  $p > 0$ , in Proposition 2.1 we show that the exactness of the complex (2) depends on the non-negative integer  $m(X)$ , the least codimension of the zero-locus of a non-zero holomorphic one-form, i.e.

$$m(X) = \min\{\text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\},$$

with the convention  $m(X) = \infty$  if every non-zero holomorphic 1-form is everywhere non-vanishing. For instance, it was observed in [GL1] Remark on p. 405 and in [La] Proposition 6.3.10, examples of varieties with  $m(X) = \dim X$  are the smooth subvarieties of an abelian variety with ample normal bundle (hence all smooth subvarieties of a simple abelian variety). In particular we show that if  $m(X) > p$  then the complex (2) is exact at the first  $(m(X) - p)$ -steps from the left with the first  $m(X) - p$  maps being of constant rank. This is enough to ensure that the cokernel of the map

$$\mathcal{O}_{\mathbf{P}}(m(X) - d - p - 1) \otimes H^{m(X)-p-1}(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{P}}(m(X) - d - p) \otimes H^{m(X)-p}(X, \Omega_X^p)$$

is a locally free sheaf. This in turn leads to inequalities for the Hodge numbers by the Evans-Griffith Theorem and by the fact that the Chern classes of a globally generated locally free sheaf are non-negative.

We turn to a more detailed presentation of our results. To simplify notation we only present the case  $m(X) = \dim X$  and we refer to Theorem 3.1 and Theorem 3.2 for general statements where all possible values of  $m(X)$  are considered. Fix an integer  $1 \leq p \leq d$  and for any  $1 \leq i \leq q - 1$  define  $\gamma_i(X, \Omega_X^p)$  to be the coefficient of  $t^i$  in the formal power series:

$$\gamma(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^p (1 - jt)^{(-1)^j h^{p, d-p+j}} \in \mathbf{Z}[[t]],$$

where  $h^{i,j} = h^{i,j}(X)$  are the Hodge numbers of  $X$ .

**Theorem 1.1.** *Let  $X$  be a compact Kähler manifold of dimension  $d$ , irregularity  $q \geq 1$  and with  $m(X) = d$ . Then*

- (i). *Any Schur polynomial of weight  $\leq q - 1$  in the  $\gamma_i(X, \Omega_X^p)$  is non-negative. In particular*

$$\gamma_i(X, \Omega_X^p) \geq 0.$$

- (ii). *If  $q > \max \{p, d - p - 1\}$  then*

$$\sum_{j=d-p}^d (-1)^{d-p+j} h^{p,j} \geq q - p.$$

For instance, the  $\gamma_1(X, \Omega_X^p)$ 's are non-negative linear polynomials in the variables  $h^{p,j}$ , and in the case of surfaces the inequalities above become  $h^{0,2} \geq 2q - 3$  and  $h^{1,1} \geq 2q - 1$ , well-known inequalities true for surfaces which do not admit any irregular pencils of genus  $\geq 2$  (cf. [BHPV] IV.5.4). In higher dimension the polynomials  $\gamma_i(X, \Omega_X^p)$ , of degree  $i$  in the variables  $h^{p,j}$ , give new inequalities involving Hodge numbers of  $X$ . In addition to the methods for  $\mathcal{O}_X$ , for  $p > 0$  Serre Duality offers a way to see until which step the complex (2) is exact *counting from the right*. This trick leads to further inequalities for the Hodge numbers and, in the special case when  $m(X) = \dim X$  and  $q(X) > \dim X \geq 2$ , also to get a bound on the Euler characteristic for the bundles  $\Omega_X^p$ :

$$|\chi(\Omega_X^1)| \geq 2 \quad \text{and} \quad |\chi(\Omega_X^p)| \geq 1 \quad \text{for} \quad p = 2, \dots, \dim X - 2$$

(cf. Corollary 2.3). In section IV we list the inequalities coming from Theorem 1.1 for manifolds of dimension three, four and five. Finally for threefolds and fourfolds with  $m(X) = \dim X$  we are able to give asymptotic bounds for all the Hodge numbers in terms of the irregularity  $q$ . In the case of threefolds we obtain

$$h^{0,2} \succeq 4q, \quad h^{0,3} \succeq 4q, \quad h^{1,1} \succeq 2q + \sqrt{2q}, \quad h^{1,2} \succeq 5q + \sqrt{2q}$$

and for the case of fourfolds we get

$$\begin{aligned} h^{0,2} \succeq 4q, \quad h^{0,3} \succeq 5q + \sqrt{2q}, \quad h^{0,4} \succeq 4q \\ h^{1,1} \succeq 2q, \quad h^{1,2} \succeq 8q + 2\sqrt{2q}, \quad h^{1,3} \succeq 12q + 3\sqrt{2q}, \quad h^{2,2} \succeq 8q + 4\sqrt{2q}. \end{aligned}$$

Asymptotic inequalities for Hodge numbers of type  $h^{0,j}$  were already established in [LP] for manifolds which do not carry any irregular fibrations.

Setting  $E = \bigwedge^* H^1(X, \mathcal{O}_X)$  for the graded exterior algebra over  $H^1(X, \mathcal{O}_X)$ , in the last section, we use the Bernstein-Gel'fand-Gel'fand (BGG) correspondence and Generic Vanishing Theorems for bundles of holomorphic  $p$ -forms to bound the *regularity* of the  $E$ -modules  $\bigoplus_i H^i(X, \Omega_X^p)$  for any value of  $p$ . We refer to Section V for the definition of regularity for finitely generated graded modules over an exterior algebra and for references about the BGG correspondence and the Generic Vanishing Theorems used. The case  $p =$

$\dim X$  has been studied in [LP] Theorem B. If we denote by  $k$  for the dimension of the general fiber of the Albanese map  $\text{alb}_X : X \rightarrow \text{Alb}(X)$ , then the  $E$ -module  $\bigoplus_i H^i(X, \omega_X)$  is  $k$ -regular but not  $(k-1)$ -regular. In general, for all the others values of  $p$ , we are not able to determine the regularity of the  $E$ -module  $\bigoplus_i H^i(X, \Omega_X^p)$  but only to give a bound in terms of the minimal and maximal dimensions of the fibers of the Albanese map  $\text{alb}_X : X \rightarrow \text{Alb}(X)$ .

**Theorem 1.2.** *Let  $X$  be a compact Kähler manifold of dimension  $d$  and irregularity  $q \geq 1$ . Let  $k$  be the dimension of the general fiber of  $\text{alb}_X : X \rightarrow \text{Alb}(X)$  and  $f$  be the maximal dimension of a fiber of  $\text{alb}_X$ . Let  $0 \leq p \leq d$  be an integer and set  $l = \max\{k, f-1\}$ . If  $p > l$  then the  $E$ -module  $\bigoplus_i H^i(X, \Omega_X^p)$  is  $(d-p+l)$ -regular.*

**Acknowledgements.** I am very grateful to Professor M. Popa for drawing my attention to this problem and for many helpful discussions. I also want to thank Professors L. Ein, R. Lazarsfeld and C. Schnell for useful conversations.

## 2. EXACTNESS OF THE COMPLEX $\underline{\mathbf{L}}_X^p$

Let  $X$  be a compact Kähler manifold of dimension  $d$ . The *irregularity* of  $X$  is the non-negative integer  $q(X) \stackrel{\text{def}}{=} h^1(X, \mathcal{O}_X)$ . The manifold  $X$  is said to be *irregular* if  $q(X) > 0$ . We aim to study the exactness of the complex (2) in terms of the non-negative integer

$$m = m(X) := \min\{\text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\},$$

with the convention  $m(X) = \infty$  if every non-zero holomorphic 1-form is nowhere vanishing. Refer also to Proposition 5.1 for another study of the exactness in terms of different invariants.

**Proposition 2.1.** *Let  $X$  be an irregular compact Kähler manifold of dimension  $d$  and  $0 \leq p \leq d$  be an integer.*

- (i). *If  $p < m \leq d$  then the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m-p$  steps from the left, and the first  $m-p$  maps are of constant rank.*
- (ii). *If  $d-p < m \leq d$  then the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m-d+p$  steps from the right, and the last  $m-d+p$  maps are of constant rank.*
- (iii). *If  $m = \infty$  then the whole complex  $\underline{\mathbf{L}}_X^p$  is exact and all the involved maps are of constant rank.*

*Proof.* Under the Hodge conjugate-linear isomorphism  $H^i(X, \Omega_X^j) \cong H^j(X, \Omega_X^i)$ , the fiber at a point  $[v] \in \mathbf{P}$  of the complex  $\underline{\mathbf{L}}_X^p$  is identified with the complex

$$(3) \quad 0 \rightarrow H^p(X, \mathcal{O}_X) \xrightarrow{\wedge \omega} H^p(X, \Omega_X^1) \xrightarrow{\wedge \omega} \dots \xrightarrow{\wedge \omega} H^p(X, \Omega_X^d) \rightarrow 0,$$

where  $\omega \in H^0(X, \Omega_X^1)$  is the holomorphic 1-form conjugate to  $v \in H^1(X, \mathcal{O}_X)$ . For every non-zero holomorphic one-form  $\omega$  the complex (3) is exact at the first  $m - p$  steps from the left by [GL1] Proposition 3.4. Hence the complex  $\underline{\mathbf{L}}_X^p$  is itself exact since exactness can be checked at the level of fibers. This also shows that the first  $m - p$  maps of the complex  $\underline{\mathbf{L}}_X^p$  are of constant rank.

For point (ii), using Serre Duality and thinking of the spaces  $H^p(X, \Omega_X^q)$  as the  $(p, q)$ -Dolbeault cohomology, we have a diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega_X^{i-1}) & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega_X^i) & \xrightarrow{\wedge\omega} & H^{d-p}(X, \Omega_X^{i+1}) & \xrightarrow{\wedge\omega} & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^p(X, \Omega_X^{d-i+1})^\vee & \longrightarrow & H^p(X, \Omega_X^{d-i})^\vee & \longrightarrow & H^p(X, \Omega_X^{d-i-1})^\vee & \longrightarrow & \dots \end{array}$$

where the bottom complex is the dual complex of (3). This diagram commutes up to sign and hence, if  $m > d - p$ , the upper complex (and therefore also the bottom one) is exact at the first  $m - d + p$  steps from the left. Finally dualizing again the bottom complex we have that (3) is exact at the first  $m - d + p$  steps from the right.

The case  $m = \infty$  follows as the complexes (3) are now everywhere exact.  $\square$

In the special case when the zero-set of every non-zero holomorphic 1-form consists of a finite number of points, i.e. when  $m(X) = \dim X$ , Proposition 2.1 implies that the complex  $\underline{\mathbf{L}}_X^p$  is everywhere exact except at most at one step. This allows us to give a bound on the Euler characteristic of the bundles of holomorphic  $p$ -forms in the case  $q(X) > \dim X$ . We recall from the introduction that examples of manifolds with  $m(X) = \dim X$  are provided by smooth subvarieties of an abelian varieties having ample normal bundle. Before stating the bounds, we prove a simple Lemma which will be useful in the sequel.

**Lemma 2.2.** *Let  $e \geq 2$ ,  $t \geq 1$ ,  $q \geq 1$  and  $a$  be integers. For  $i = 1, \dots, e + 1$  and  $s = 1, \dots, t$  let  $V_i$  and  $Z_s$  be complex vector spaces of positive dimension.*

(i). *If a complex of locally free sheaves on  $\mathbf{P} = \mathbf{P}^{q-1}$  of length  $e + 1$  of the form*

$$(4) \quad 0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a+1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a+e) \longrightarrow 0$$

*is exact, then  $q \leq e$ .*

(ii). *Let  $k_s \geq -a + e$  be integers. If a complex of locally free sheaves on  $\mathbf{P} = \mathbf{P}^{q-1}$  of length  $e + 2$  of the form*

$$0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-a) \longrightarrow V_e \otimes \mathcal{O}_{\mathbf{P}}(-a+1) \longrightarrow \dots \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}}(-a+e) \longrightarrow \bigoplus_{s=1}^t (Z_s \otimes \mathcal{O}_{\mathbf{P}}(k_s)) \longrightarrow 0$$

*is exact, then  $q \leq e + 1$ .*

*Proof.* If  $q = 1$  then clearly  $q \leq e$ , therefore we can assume  $q > 1$ . If  $e = 2$  then  $q = 2$ , since line bundles on projective spaces have no intermediate cohomology and so we can suppose

$e > 2$ . After having twisted the complex (4) by  $\mathcal{O}_{\mathbf{P}}(-e + a)$  we get the complex

$$\begin{aligned} 0 \longrightarrow V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e) &\xrightarrow{f_1} V_e \otimes \mathcal{O}_{\mathbf{P}}(-e+1) \longrightarrow \dots \\ \dots \longrightarrow V_4 \otimes \mathcal{O}_{\mathbf{P}}(-3) &\xrightarrow{f_{e-2}} V_3 \otimes \mathcal{O}_{\mathbf{P}}(-2) \longrightarrow V_2 \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V_1 \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow 0. \end{aligned}$$

Set  $W_j = \text{coker } f_j$  for  $j = 1, \dots, e-2$ . If  $q > e$ , we would have that  $H^{e-1-j}(\mathbf{P}, W_j) \neq 0$  for every  $j = 1, \dots, e-2$  and hence that  $H^{e-1}(\mathbf{P}, V_{e+1} \otimes \mathcal{O}_{\mathbf{P}}(-e)) \neq 0$ . This yields a contradiction and then  $q \leq e$ . To prove (ii) we can use the same argument used to prove (i).  $\square$

**Corollary 2.3.** *Let  $X$  be a compact Kähler manifold of dimension  $d \geq 2$  and irregularity  $q(X) > d$ . If  $m(X) = d$  then*

$$(-1)^{d-1} \chi(\Omega_X^1) \geq 2,$$

and

$$(-1)^{d-p} \chi(\Omega_X^p) \geq 1$$

for any  $p = 2, \dots, d-2$ .

*Proof.* To begin with, we note that  $h^d(X, \Omega_X^p) \neq 0$  so that the complex  $\underline{\mathbf{L}}_X^p$  is non-zero. By Proposition 2.1 (ii), the assumption  $m(X) = d$  implies that the non-zero complex  $\underline{\mathbf{L}}_X^d$  is exact at the first  $d$  steps from the right. If we had  $h^d(X, \Omega_X^p) = h^p(X, \omega_X) = 0$ , then  $\underline{\mathbf{L}}_X^d$  would induce an exact complex of length  $\leq d$  whose terms are sums of line bundles all of the same degree, and by Lemma 2.2 we would have a contradiction.

By Proposition 2.1 the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $d-p$  steps from the left and at the first  $p$  steps from the right. Therefore we get two exact sequences:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^p) &\longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-p-1) \otimes H^{d-p-1}(X, \Omega_X^p) \xrightarrow{f} \\ &\xrightarrow{f} \mathcal{O}_{\mathbf{P}}(-p) \otimes H^{d-p}(X, \Omega_X^p) \longrightarrow F \longrightarrow 0, \end{aligned}$$

where the locally free sheaf  $F$  is the cokernel of the map  $f$ , and

$$\begin{aligned} 0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(-p) \otimes H^{d-p}(X, \Omega_X^p) &\xrightarrow{g} \\ &\xrightarrow{g} \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{d-p+1}(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0, \end{aligned}$$

where the locally free sheaf  $G$  is the kernel of the map  $g$ . We also get an induced map of locally free sheaves  $h : F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{d-p+1}(X, \Omega_X^p)$ , which is of constant rank. Denoting by  $E$  the kernel of  $h$  we obtain a new exact sequence of locally free sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_{\mathbf{P}}(-p+1) \otimes H^{d-p+1}(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0$$

from which we can read  $\text{rank } E = (-1)^{d-p} \chi(\Omega_X^p)$ . If the locally free sheaf  $E$  were the zero sheaf then the complex  $\underline{\mathbf{L}}_X^p$  would be an exact complex of length  $\leq d+1$  whose terms are sum of line bundles all of the same degree, which is impossible by Lemma 2.2 (i). Thus

$$\text{rank } E = (-1)^{d-p} \chi(\Omega_X^p) \geq 1.$$

If  $p = d - 1$  we can improve our bound. In this case the complex  $\underline{\mathbf{L}}_X^{d-1}$  is exact at the first  $d - 1$  steps from the right, and hence we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^{d-1}) \xrightarrow{h'} G \longrightarrow E' \longrightarrow 0,$$

where  $E'$  is the cokernel of the map  $h'$ . The locally free sheaf  $E'$  is non-zero again by Lemma 2.2. If the rank of  $E'$  were one, then  $E'$  would be a line bundle,  $E' = \mathcal{O}_{\mathbf{P}}(k)$  for some integer  $k$ , and  $G \in \text{Ext}^1(\mathcal{O}_{\mathbf{P}}(k), \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^{d-1})) = H^1(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(d - k) \otimes H^0(X, \Omega_X^{d-1})^\vee) = 0$ . Hence  $G$  would split as a sum of line bundles and by Lemma 2.2 (ii) this is not possible. Therefore

$$\text{rank } E' = (-1)^{d-1} \chi(\Omega_X^1) \geq 2.$$

□

### 3. INEQUALITIES FOR THE HODGE NUMBERS

After having studied the exactness of the complex (2) we can derive inequalities for the Hodge numbers by using well-known results for locally free sheaves on projective spaces: the Evans-Griffith Theorem and the non negativity of the Chern classes for globally generated locally free sheaves.

Throughout this section we fix integers  $d \geq 1$ ,  $q \geq 1$ ,  $0 \leq p \leq d$  and  $0 < m \leq d$ . We denote by  $h^{p,q} = h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$  the Hodge numbers of  $X$  and by  $q = q(X)$  the irregularity of  $X$ .

Before stating the results we need to introduce some notation. If  $d - p < m \leq d$ , for  $1 \leq i \leq q - 1$  we define  $\gamma_i(X, \Omega_X^p)$  to be the coefficient of  $t^i$  in the formal power series:

$$\gamma(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-d+p} (1 - jt)^{(-1)^j h^{p, 2d-m-p+j}} \in \mathbf{Z}[[t]].$$

If  $p < m \leq d$ , for  $1 \leq i \leq q - 1$  we define  $\delta_i(X, \Omega_X^p)$  to be the coefficient of  $t^i$  in the formal power series:

$$\delta(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^{m-p} (1 - jt)^{(-1)^j h^{p, m-p-j}} \in \mathbf{Z}[[t]].$$

If  $m = \infty$ , for  $i = 1, \dots, q - 1$  we define  $\varepsilon_i(X, \Omega_X^p)$  to be the coefficient of  $t^i$  in the formal power series:

$$\varepsilon(X, \Omega_X^p; t) \stackrel{\text{def}}{=} \prod_{j=1}^d (1 - jt)^{(-1)^j h^{p, d-j}} \in \mathbf{Z}[[t]].$$

Also consider the following pieces of the Euler characteristic of the bundle  $\Omega_X^p$ . If  $d - p < m \leq d$  define

$$\chi^{\geq 2d-m-p}(\Omega_X^p) \stackrel{\text{def}}{=} \sum_{j=2d-m-p}^d (-1)^{2d-m-p+j} h^{p,j}$$

and if  $p < m \leq d$  define

$$\chi^{\leq m-p}(\Omega_X^p) \stackrel{\text{def}}{=} \sum_{j=0}^{m-p} (-1)^{m-p+j} h^{p,j}.$$

**Theorem 3.1.** *Let  $X$  be a compact Kähler manifold of dimension  $d$  and irregularity  $q \geq 1$ . Let  $m = m(X) = \min\{\text{codim } Z(\omega) \mid 0 \neq \omega \in H^0(X, \Omega_X^1)\}$  and let  $0 \leq p \leq d$  be an integer.*

- (i). *If  $d - p < m \leq d$  then any Schur polynomial of weight  $\leq q - 1$  in the  $\gamma_i(X, \Omega_X^p)$  is non-negative. In particular*

$$\gamma_i(X, \Omega_X^p) \geq 0$$

*for every  $1 \leq i \leq q - 1$ . Moreover, if  $i$  is an index with  $\chi^{\geq 2d-m-p}(\Omega_X^p) < i < q$ , then  $\gamma_i(X, \Omega_X^p) = 0$ .*

- (ii). *If  $p < m \leq d$  then any Schur polynomial of weight  $\leq q - 1$  in the  $\delta_i(X, \Omega_X^p)$  is non-negative. In particular*

$$\delta_i(X, \Omega_X^p) \geq 0$$

*for every  $1 \leq i \leq q - 1$ . Moreover, if  $i$  is an index with  $\chi^{\leq m-p}(\Omega_X^p) < i < q$ , then  $\delta_i(X, \Omega_X^p) = 0$ .*

- (iii). *If  $m = \infty$  then*

$$\varepsilon_i(X, \Omega_X^p) = 0$$

*for every  $i = 1, \dots, q - 1$ .*

*Proof.* If  $m > d - p$  then by Proposition 2.1 (ii) the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m - d + p$  steps from the right, and hence we get the exact sequence

$$(5) \quad 0 \longrightarrow G \longrightarrow \mathcal{O}_{\mathbf{P}}(d - m - p) \otimes H^{2d-m-p}(X, \Omega_X^p) \xrightarrow{g} \\ \xrightarrow{g} \mathcal{O}_{\mathbf{P}}(d - m - p + 1) \otimes H^{2d-m-p+1}(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0,$$

where  $G$  is the kernel of the map  $g$ . Tensoring (5) by  $\mathcal{O}_{\mathbf{P}}(d - m - p)$  and then dualizing it, we note that the polynomial  $\gamma(X, \Omega_X^p; t)$  is the Chern polynomial of the locally free sheaf  $G^\vee(m - d + p)$ . Then its Chern classes  $c_i(G^\vee(m - d + p))$ , as well as the Schur polynomials in these, are non-negative since  $G^\vee(m - d + p)$  is globally generated. In particular we get

$$\gamma_i(X, \Omega_X^p) = \deg c_i(G^\vee(m - d + p)) \geq 0.$$

The last statement of (i) follows from the fact that  $c_i(G) = 0$  for  $i > \text{rank } G = \chi^{\geq 2d-m-p}(\Omega_X^p)$ .

The proof of (ii) is analogous to the proof of the previous point. If  $m > p$  then by Proposition 2.1 (i) the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m - p$  steps from the left and induces the following exact complex

$$(6) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-d + m - p - 1) \otimes H^{m-p-1}(X, \Omega_X^p) \xrightarrow{f} \\ \xrightarrow{f} \mathcal{O}_{\mathbf{P}}(-d + m - p) \otimes H^{m-p}(X, \Omega_X^p) \longrightarrow F \longrightarrow 0$$



where  $F$  is the cokernel of the map  $f$ . Tensoring (6) by  $\mathcal{O}_{\mathbf{P}}(d-m+p)$  we get that the locally free sheaf  $F(d-m+p)$  is globally generated and moreover that its Chern polynomial is the polynomial  $\delta(X, \Omega_X^p; t)$ . Now we conclude as in (i).

If  $m = \infty$  then the complex  $\underline{\mathbf{L}}_X^p$  is everywhere exact and the polynomial  $\varepsilon(X, \Omega_X^p; t)$  is just the Chern polynomial of the zero sheaf. Thus its Chern classes satisfy  $\varepsilon_i(X, \Omega_X^p) = 0$ , for every  $i = 1, \dots, q-1$ .  $\square$

Under the assumption of Theorem 3.1 we also have

**Theorem 3.2.** (i). *If  $d-p < m \leq d$  and  $q > \max\{m-d+p, d-p-1\}$  then*

$$\chi^{\geq 2d-m-p}(\Omega_X^p) \geq q + d - m - p$$

and

$$h^{d-p,1} \geq h^{d-p,0} + q - 1.$$

(ii). *If  $p < m \leq d$  and  $q > \max\{m-p, p-1\}$  then*

$$\chi^{\leq m-p}(\Omega_X^p) \geq q - m + p$$

and

$$h^{p,1} \geq h^{p,0} + q - 1.$$

*Proof.* (i). By Proposition 2.1 the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m-d+p$  steps from the right. Since  $q > d-p-1$  we can prove, with an argument similar to the one used in Corollary 2.3, that  $h^d(X, \Omega_X^p)$  is non-zero and hence that the complex  $\underline{\mathbf{L}}_X^p$  is non-zero as well (this observation allow us to use Lemma 2.2). If  $q = m-d+p+1$  then it is enough to prove that the rank of the locally free sheaf  $G$  in (5) is at least one. By Lemma 2.2 the locally free sheaf  $G$  is neither zero nor splits as a sum of line bundles. Then by the Evans-Griffith Theorem (see [La] p. 92)

$$\text{rank } G = \chi^{\geq 2d-m-p}(\Omega_X^p) \geq q + d - m - p.$$

For the inequality  $h^{d-p,1} \geq h^{d-p,0} + q - 1$ , it is enough to note that the complex  $\underline{\mathbf{L}}_X^p$  induces a surjection  $H^{d-1}(X, \Omega_X^p) \otimes \mathcal{O}_{\mathbf{P}} \rightarrow H^d(X, \Omega_X^p) \otimes \mathcal{O}_{\mathbf{P}}(1) \rightarrow 0$  and, since  $h^d(X, \Omega_X^p) \neq 0$ , we can conclude thanks to Example 7.2.2 in [La].

(ii). The hypothesis  $p < m \leq d$  implies that the complex  $\underline{\mathbf{L}}_X^p$  is exact at the first  $m-p$  steps from the left. Since  $q > p-1$  we have that  $h^0(X, \Omega_X^p) = h^d(X, \Omega_X^{d-p}) \neq 0$  as in (i), and therefore that the complex  $\underline{\mathbf{L}}_X^p$  is non-zero as well. After having noted that  $\text{rank } F = \chi^{\leq m-p}(\Omega_X^p)$  we can argue as in the previous point.  $\square$

**3.1. The case  $m(X) = \dim X$ .** When  $X$  is an irregular compact Kähler manifold with  $m(X) = \dim X$  further inequalities hold thanks to Catanese's work [Cat]. Let  $\text{alb}_X : X \rightarrow \text{Alb}(X)$  be the Albanese map of  $X$ . We say that  $X$  is of *maximal Albanese dimension* if  $\dim \text{alb}_X(X) = \dim X$ . Following Catanese's terminology we say that  $X$  is of *Albanese general type* if  $q(X) > \dim X$  and if it is of maximal Albanese dimension. A *higher irrational pencil* is a surjective map with connected fibers  $f : X \rightarrow Y$  onto a normal lower dimensional variety  $Y$  and such that any smooth model of  $Y$  is of Albanese general type.

**Lemma 3.3.** *If  $X$  is an irregular compact Kähler manifold with  $m(X) = \dim X$ , then  $X$  does not carry any higher irrational pencils.*

*Proof.* We proceed by contradiction. Suppose a higher irrational pencil  $f : X \rightarrow Y$  exists and let  $\text{alb}_Y : Y \rightarrow \text{Alb}(Y)$  be the Albanese map of  $Y$ , which is well defined since  $Y$  is normal. The map  $\text{alb}_Y$  is not surjective, hence following an idea contained in the proof of [EL] Proposition 2.2, one can show that given a general point  $y \in Y$  there exists a holomorphic 1-form  $\omega$  of  $\text{Alb}(Y)$  whose restriction to  $\text{alb}_Y(Y)$  vanishes at  $\text{alb}_Y(y)$ . Pulling back  $\omega$  to  $X$ , we get a holomorphic 1-form which vanishes along some fibers of  $f$  which are of positive dimension, this contradicting the hypothesis  $m(X) = \dim X$ . The form  $\omega$  can be constructed as follows. Let  $z$  be a smooth point of the Albanese image  $\text{alb}_Y(Y) \subset \text{Alb}(Y)$ . The coderivative map  $T_z^* \text{Alb}(Y) \rightarrow T_z^* \text{alb}_Y(Y)$  is surjective with non trivial kernel. Then take  $\omega$  to be the extension to a holomorphic 1-form on  $\text{Alb}(Y)$  of any non-zero form belonging to this kernel.  $\square$

In the following Proposition we collect inequalities for Hodge numbers in the case  $m(X) = \dim X$ , which will be used to give asymptotic bounds for Hodge numbers in terms of the irregularity for manifolds of dimension three and four (*cf.* Corollary 4.1 and Corollary 4.2). These are essentially extracted from [LP] Remark 3.3 and the references therein together with Lemma 3.3.

**Proposition 3.4.** *Let  $X$  be an irregular compact Kähler manifold with  $m(X) = \dim X$ . Then*

$$(7) \quad h^{0,k} \geq k(q(X) - k) + 1$$

*for any  $k = 0, \dots, \dim X$ . If  $\dim X \geq 3$  then*

$$(8) \quad h^{0,2} \geq 4q(X) - 10.$$

*If  $q(X) \geq \dim X$ , and for any value of  $\dim X$ , then*

$$(9) \quad \chi(\omega_X) \geq q(X) - \dim X.$$

*Proof.* One can show that if  $\omega_1, \dots, \omega_k$  are linearly independent holomorphic 1-forms then  $\omega_1 \wedge \dots \wedge \omega_k \neq 0$ . This can be done by induction on  $k$  and by using the above Lemma 3.3, Theorem 1.14 and Lemma 2.20 in [Cat]. This fact can be reformulated as saying that the natural maps

$$\phi_k : \bigwedge^k H^1(X, \mathcal{O}_X) \rightarrow H^k(X, \mathcal{O}_X)$$

are injective on primitive forms  $\omega_1 \wedge \dots \wedge \omega_k$ . The set of such forms is the cone over the image of the Plücker embedding, i.e. over the Grassmannian  $\mathbf{G}(k, H^1(X, \mathcal{O}_X))$ , and by comparing the dimensions we have the bounds. For the inequality (8) we can apply the same argument used in [LP] Remark 3.3 and for the inequality (9) we note that when  $q(X) \geq \dim X$  then  $X$  is of maximal Albanese dimension. In fact by [GL1] Remark on p. 405, the cohomological support loci  $V^i(\omega_X) = \{\alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0\}$  consist of at most a finite set of points and, by [LP] Remark 1.4 or by [BLNP] Proposition 2.7,  $X$  is of maximal Albanese dimension. Now the inequality  $\chi(\omega_X) \geq q(X) - \dim X$  follows by [PP2] Corollary 4.2 or by [LP] Theorem 3.1.  $\square$

#### 4. EXAMPLES AND ASYMPTOTIC BOUNDS FOR THREEFOLDS AND FOURFOLDS

In this section we list concrete inequalities coming from Theorems 3.1 and 3.2 in the most interesting case  $m(X) = \dim X$ ,  $q(X) \geq \dim X$ , and for  $\dim X = 3, 4, 5$ . Moreover for threefolds and fourfolds we list asymptotic bounds in terms of the irregularity  $q(X)$  for all the Hodge numbers. We also point out that some of the inequalities are still valid for some values of  $q(X)$  smaller than  $\dim X$  and that other inequalities hold for different values of  $m(X)$ . Set  $q = q(X)$  for the irregularity and  $h^{p,q} = h^{p,q}(X)$  for the Hodge numbers.

Let us start with Theorem 3.1. We get a first set of inequalities by imposing the conditions  $\gamma_1(X, \Omega_X^k) \geq 0$  for  $k = 0, 1, 2$ . Hence

$$\begin{aligned} h^{0,2} &\geq 2q - 3, & h^{1,1} &\geq 2q && \text{for } \dim X = 3 \\ h^{1,2} &\geq 2h^{1,1} - 3q, & h^{1,2} &\geq 2h^{0,2}, & h^{0,3} &\geq 2h^{0,2} - 3q + 4 && \text{for } \dim X = 4 \\ h^{0,4} &\geq 4q - 3h^{0,2} + 2h^{0,3} - 5, & h^{1,4} &\geq 4h^{1,1} - 3h^{1,2} + 2h^{1,3}, & h^{2,2} &\geq 2h^{1,2} - 3h^{0,2} && \text{for } \dim X = 5 \end{aligned}$$

Finer inequalities are obtained by solving  $\gamma_2(X, \Omega_X^k) \geq 0$ . Then for  $\dim X = 3$  we have

$$(10) \quad h^{0,2} \geq 2q - \frac{7}{2} + \frac{\sqrt{8q-23}}{2}, \quad h^{1,1} \geq 2q - \frac{1}{2} + \frac{\sqrt{8q+1}}{2}$$

and for  $\dim X = 4$  we get

$$(11) \quad h^{0,3} \geq 2h^{0,2} - 3q + \frac{7}{2} + \frac{\sqrt{8h^{0,2} - 24q + 49}}{2}$$

$$(12) \quad h^{1,2} \geq 2h^{1,1} - 3q + \sqrt{4h^{1,1} - 9q}$$

$$(13) \quad h^{1,2} \geq 2h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{0,2} + 1}}{2}$$

where the quantity  $4h^{1,1} - 9q$  is non-negative by the second inequality of Theorem 3.2 (i) and the quantity  $8h^{0,2} - 24q + 49$  is non-negative by inequality (8). Finally for  $\dim X = 5$

we get

$$\begin{aligned} h^{0,4} &\geq 4q - 3h^{0,2} + 2h^{0,3} - \frac{11}{2} + \frac{\sqrt{48q - 24h^{0,2} + 8h^{0,3} - 79}}{2} \\ h^{1,4} &\geq 2h^{1,3} + 4h^{1,1} - 3h^{1,2} - \frac{1}{2} + \frac{\sqrt{48h^{1,1} - 24h^{1,2} + 8h^{1,3} + 1}}{2} \\ h^{2,2} &\geq 2h^{1,2} - 3h^{0,2} - \frac{1}{2} + \frac{\sqrt{8h^{1,2} - 24h^{0,2} + 1}}{2} \end{aligned}$$

which hold as long as the quantity under the square root are non-negative.

Applying Theorem 3.2 with  $m(X) = \dim X$  and  $q(X) \geq \dim X$ , we get for  $\dim X = 3$

$$\chi(\omega_X) \geq q - 3, \quad h^{1,1} \geq 2q - 1, \quad h^{1,2} \geq h^{1,1} - 2, \quad h^{1,2} \geq h^{0,2} + q - 1,$$

for  $\dim X = 4$

$$\begin{aligned} \chi(\omega_X) &\geq q - 4, \quad h^{2,2} \geq h^{1,2} - h^{0,2} + q - 2, \quad h^{1,3} \geq h^{1,2} - h^{1,1} + 2q - 3 \\ h^{1,1} &\geq 2q - 1, \quad h^{1,2} \geq h^{2,0} + q - 1, \quad h^{1,3} \geq h^{0,3} + q - 1 \end{aligned}$$

and for  $\dim X = 5$

$$\begin{aligned} \chi(\omega_X) &\geq q - 5, \quad h^{1,4} \geq h^{1,3} - h^{1,2} + h^{1,1} - 4, \quad h^{1,1} \geq 2q - 1, \\ 2h^{1,2} &\geq h^{2,2} + h^{0,2} + q - 3, \quad h^{1,2} \geq h^{0,2} + q - 1, \quad h^{1,2} \geq h^{1,3} - h^{0,3} + q - 2, \\ h^{1,3} &\geq h^{0,3} + q - 1, \quad h^{1,4} \geq h^{0,4} + q - 1. \end{aligned}$$

We select the strongest of the inequalities above in dimension three and four in statements formulated asymptotically for simplicity:

**Corollary 4.1.** *Let  $X$  be an irregular compact Kähler threefold with  $m(X) = 3$ . Then asymptotically*

$$h^{0,2} \succeq 4q, \quad h^{0,3} \succeq 4q, \quad h^{1,1} \succeq 2q + \sqrt{2q}, \quad h^{1,2} \succeq 5q + \sqrt{2q}.$$

*Proof.* The inequality (8) gives  $h^{0,2} \succeq 4q$ . The inequality  $\chi(\omega_X) \geq q - 3$  of Theorem 3.2 implies the inequality  $h^{0,3} \geq h^{0,2} - 2$  and therefore asymptotically  $h^{0,3} \succeq 4q$ . The asymptotic bound for  $h^{1,1}$  follows by (10). Since by Corollary 2.3  $\chi(\Omega_X^1) \geq 2$  we also get the bound for  $h^{1,2}$ .  $\square$

**Corollary 4.2.** *Let  $X$  be an irregular compact Kähler fourfold with  $m(X) = 4$ . Then asymptotically*

$$\begin{aligned} h^{0,2} &\succeq 4q, \quad h^{0,3} \succeq 5q + \sqrt{2q}, \quad h^{0,4} \succeq 4q \\ h^{1,1} &\succeq 2q, \quad h^{1,2} \succeq 8q + 2\sqrt{2q}, \quad h^{1,3} \succeq 12q + 3\sqrt{2q}, \quad h^{2,2} \succeq 8q + 4\sqrt{2q}. \end{aligned}$$

*Proof.* The asymptotic bounds for  $h^{0,2}$ ,  $h^{0,3}$  and  $h^{0,4}$  follow by (8), (11) and (7) respectively. Using the second inequality of Theorem 3.2 (i) we get  $h^{1,1} \succeq 2q$ , and by inequality (13) we get the bound for  $h^{1,2}$ . By Corollary 2.3 we have  $\chi(\Omega_X^1) \leq 2$  and  $\chi(\Omega_X^2) \geq 1$  which imply the bounds for  $h^{1,3}$  and  $h^{2,2}$ .  $\square$

## 5. REGULARITY OF THE COHOMOLOGY MODULES

In this section we give the proof of Theorem 1.2 from the Introduction.

Let  $V$  be a complex vector space and  $E = \bigwedge^* V$  be the graded exterior algebra over  $V$ . A finitely generated graded  $E$ -module  $P = \bigoplus_{j \geq 0} P_j$  is called *m-regular* if it is generated in degrees 0 up to  $-m$ , and if its minimal free resolution has at most  $m + 1$  linear strands. Equivalently,  $P$  is *m-regular* if and only if  $\text{Tor}_i^E(P, \mathbf{C})_{-i-j} = 0$  for all  $i \geq 0$  and all  $j \geq m + 1$ . The *dual* over  $E$  of the module  $P$  is defined to be the  $E$ -module  $Q = \hat{P} = \bigoplus_j P_{-j}^*$  (cf. [Eis], [EFS], [LP]).

We continue to denote by  $X$  an irregular compact Kähler manifold of dimension  $d$  and irregularity  $q$ . Set  $V = H^1(X, \mathcal{O}_X)$  and  $E = \bigwedge^* V$ . In [LP] the authors determined the regularity of the graded  $E$ -module  $Q_X = \bigoplus_i H^i(X, \omega_X)$  by studying the exactness of the complex associated to its dual module  $P_X = \bigoplus_i H^i(X, \mathcal{O}_X)$  via the BGG correspondence. For references on the BGG correspondence see [BGG], [EFS] and Chapter 7B of [Eis]. By applying their same technique and by using Generic Vanishing Theorems for bundles of holomorphic  $p$ -forms (see [PP1] and [CH]) we give a bound for the regularity of the  $E$ -modules  $\bigoplus_i H^i(X, \Omega_X^p)$  for any  $p$ .

Fix an integer  $p = 0, \dots, d$ . Via cup product consider the graded  $E$ -module

$$P_X^p = \bigoplus_i H^i(X, \Omega_X^{d-p})$$

where the graded piece  $H^i(X, \Omega_X^{d-p})$  has degree  $d - i$ . The dual module over  $E$  of  $P_X^p$  is the module

$$Q_X^p = \bigoplus_i H^i(X, \Omega_X^p)$$

where the graded piece  $H^i(X, \Omega_X^p)$  has degree  $-i$ .

Let  $W = V^*$  be the dual vector space to  $V$  and  $S = \text{Sym}(W)$  be the symmetric algebra over  $W$ . Also denote by  $\mathbf{P} = \mathbf{P}_{\text{sub}}(V)$  the projective space of dimension  $q - 1$  over  $V$ . By applying the functor  $\Gamma_*$  to the complex  $\underline{\mathbf{L}}_X^p$ , we get the complex  $\mathbf{L}_X^p$  of  $S$ -graded modules in homological degree 0 to  $d$

$$\mathbf{L}_X^p : 0 \longrightarrow S \otimes_{\mathbf{C}} H^0(X, \Omega_X^p) \longrightarrow S \otimes_{\mathbf{C}} H^1(X, \Omega_X^p) \longrightarrow \dots \longrightarrow S \otimes_{\mathbf{C}} H^d(X, \Omega_X^p) \longrightarrow 0.$$

(see [Ha] p. 118 for the definition of  $\Gamma_*$ ). To bound the regularity of the module  $Q_X^p = \bigoplus_i H^i(X, \Omega_X^p)$  is enough to understand until which step the complex  $\mathbf{L}_X^{d-p}$  is exact. In fact, by following the proof of [LP] Lemma 2.3, one can show that the complex  $\mathbf{L}_X^{d-p}$  is identified with the complex associated to the module  $P_X^p = \bigoplus_i H^i(X, \Omega_X^{d-p})$  via the BGG correspondence and one can use Proposition 2.2 (cf. *loc. cit.*) to bound the regularity of  $Q_X^p$ . At this point Theorem 1.2 follows by the previous discussion together with point (i) of the following Proposition.

**Proposition 5.1.** *Let  $X$  be an irregular compact Kähler manifold of dimension  $d$ . Let  $k$  be the dimension of the generic fiber of the Albanese map  $\text{alb}_X : X \longrightarrow \text{Alb}(X)$  and let  $f$  be the maximal dimension of a fiber of  $\text{alb}_X$ . Set  $l = \max\{k, f - 1\}$ .*

- (i). If  $d - p > l$  then the complexes  $\mathbf{L}_X^p$  and  $\underline{\mathbf{L}}_X^p$  are exact in the first  $d - p - l$  steps from the left.
- (ii). If  $p > l$  then the complexes  $\mathbf{L}_X^p$  and  $\underline{\mathbf{L}}_X^p$  are exact in the first  $p - l$  steps from the right.

*Proof.* We follow [LP] Proposition 1.1. Let  $\mathbf{A} = \text{Spec}(\text{Sym}(W))$  be the affine space corresponding to  $V$  viewed as an affine algebraic variety. Consider the following complex  $\mathcal{K}_p$  of trivial locally free sheaves on  $\mathbf{A}$ :

$$0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0,$$

with maps given at each point of  $\mathbf{A}$  by cupping with the corresponding element of  $V$ . Since  $\Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}}) = S$  one sees that

$$(14) \quad \mathbf{L}_X^p = \Gamma(\mathbf{A}, \mathcal{K}_p)$$

i.e.  $\mathbf{L}_X^p$  is obtained from  $\mathcal{K}_p$  by applying the global sections functor. On the other hand the complex  $\underline{\mathbf{L}}_X^p$  is obtained by  $\mathbf{L}_X^p$  via sheafification which is an exact functor, and hence its exactness is implied by the exactness of the complex  $\mathbf{L}_X^p$ , and consequently by the exactness of  $\mathcal{K}_p$  since the functor global sections is exact on affine spaces. Let  $\mathbf{V}$  be the vector space  $V$  considered as a complex manifold. By GAGA, the exactness of  $\mathcal{K}_p$  is equivalent to the exactness of  $\mathcal{K}_p^{\text{an}}$ , where  $\mathcal{K}_p^{\text{an}}$  is the complex

$$0 \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^0(X, \Omega_X^p) \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^1(X, \Omega_X^p) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^d(X, \Omega_X^p) \longrightarrow 0$$

of analytic sheaves on  $\mathbf{V}$ . If  $d - p > l$  then it is enough to check the exactness of  $\mathcal{K}_p^{\text{an}}$  at the first  $d - p - l$  steps from the left, or equivalently the vanishing of the cohomologies  $\mathcal{H}^i(\mathcal{K}_p^{\text{an}})$  for any  $i < d - p - l$ . Since the differentials of the complex  $\mathcal{K}_p^{\text{an}}$  scale linearly in radial directions through the origin, it is then enough to check the vanishing of its stalks at the origin 0, i.e.

$$\mathcal{H}^i(\mathcal{K}_p^{\text{an}})_0 = 0$$

for  $i < d - p - l$ .

Let now  $p_1 : X \times \text{Pic}^0(X) \longrightarrow X$  and  $p_2 : X \times \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$  be the projections onto the first and second factor respectively and let  $\mathcal{P}$  be a normalized Poincaré line bundle on  $X \times \text{Pic}^0(X)$ . Then Theorem 6.2 in [CH] gives an identification

$$\mathcal{H}^i(\mathcal{K}_p^{\text{an}})_0 \cong R^i p_{2*}(p_1^* \Omega_X^p \otimes \mathcal{P})_0,$$

via the exponential map  $\exp : V \longrightarrow \text{Pic}^0(X)$ . At this point Theorem 5.11 (1) and Theorem 3.7 in [PP1] imply that  $R^i p_{2*}(p_1^* \Omega_X^p \otimes \mathcal{P}) = 0$  for any  $i < d - p - l$ .

The proof of (ii) is analogous to the proof in Proposition 2.1. □

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